

Shirakawa, K.  
Osaka J. Math.  
50 (2013), 53–60

# GLOBAL MONODROMY MODULO 5 OF THE QUINTIC-MIRROR FAMILY

KENNICHIRO SHIRAKAWA

(Received December 10, 2010, revised May 10, 2011)

## Abstract

The quintic-mirror family is a well-known one-parameter family of Calabi–Yau threefolds. A complete description of the global monodromy group of this family is not yet known. In this paper, we give a presentation of the global monodromy group in the general linear group of degree 4 over the ring of integers modulo 5.

## 1. Introduction

The quintic-mirror family  $(W_\lambda)_{\lambda \in \mathbb{P}^1} \rightarrow \mathbb{P}^1$  is a family, whose restriction  $f: (W_\lambda)_{\lambda \in U} \rightarrow U$  on  $U := \mathbb{P}^1 - \{0, 1, \infty\}$  is a smooth projective family of Calabi–Yau manifolds. Fix  $b \in U$  and let  $\langle \cdot, \cdot \rangle$  be the anti-symmetric bilinear form on  $H^3(W_b, \mathbb{Z})$  defined by the cup product. The global monodromy group  $\Gamma$  is the image of the representation  $\pi_1(U, b) \rightarrow \text{Aut}(H^3(W_b, \mathbb{Z}), \langle \cdot, \cdot \rangle)$  corresponding to the local system  $R^3 f_* \mathbb{Z}$  with the fiber  $H^3(W_b, \mathbb{Z})$  over  $b$ . When we take a symplectic basis, we can identify  $\text{Aut}(H^3(W_b, \mathbb{Z}), \langle \cdot, \cdot \rangle)$  with  $\text{Sp}(4, \mathbb{Z})$ .

In this paper, we are concerned with a description of  $\Gamma$ . Matrix presentations of the generators of  $\Gamma$  are well studied and it is also known that  $\Gamma$  is Zariski dense in  $\text{Sp}(4, \mathbb{Z})$  (e.g. [1], [3]). However, it is not known whether the index of  $\Gamma$  in  $\text{Sp}(4, \mathbb{Z})$  is finite or not (e.g. [2]). A direct approach for this problem is to describe  $\Gamma$  explicitly. In the main theorem of this paper, we give a presentation of  $\Gamma$  in  $\text{GL}(4, \mathbb{Z}/5\mathbb{Z})$ , which is a small attempt toward a description of  $\Gamma$ .

On the other hand, Chen, Yang and Yui find a congruence subgroup  $\Gamma(5, 5)$  of  $\text{Sp}(4, \mathbb{Z})$  of finite index, which contains  $\Gamma$  in [2]. Combining their result and our main theorem, we can construct a smaller congruence subgroup  $\tilde{\Gamma}(5, 5)$  of  $\text{Sp}(4, \mathbb{Z})$  of finite index, which contains  $\Gamma$ . However this result is merely the fact that  $\tilde{\Gamma}(5, 5)$  contains  $\Gamma$ . After all, the index of  $\Gamma$  in  $\text{Sp}(4, \mathbb{Z})$  is still unknown.

## 2. The quintic-mirror family

The quintic-mirror family was constructed by Greene and Plesser. We review the construction of the quintic-mirror family after [4].

Let  $\psi \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ , and let

$$Q_\psi = \{x \in \mathbb{P}^4 \mid x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - 5\psi x_1 x_2 x_3 x_4 x_5 = 0\}.$$

A finite group  $G$ , which is abstractly isomorphic to  $(\mathbb{Z}/5\mathbb{Z})^3$ , acts on  $Q_\psi$  as follows.

$\mu_5$ : the multiplicative group of the 5-th root of 1  $\in \mathbb{C}$ ,

$$\tilde{G} = (\mu_5)^5 / \{(\alpha_1, \dots, \alpha_5) \in (\mu_5)^5 \mid \alpha_1 = \dots = \alpha_5\},$$

$$G = \{(\alpha_1, \dots, \alpha_5) \in \tilde{G} \mid \alpha_1 \cdots \alpha_5 = 1\},$$

$$G \times Q_\psi \rightarrow Q_\psi, \quad ((\alpha_1, \dots, \alpha_5), (x_1, \dots, x_5)) \mapsto (\alpha_1 x_1, \dots, \alpha_5 x_5).$$

When we take the quotient of the hypersurface  $Q_\psi$  by  $G$ , canonical singularities appear. For  $\psi \in \mathbb{C} \subset \mathbb{P}^1$ , it is known that there is a simultaneous minimal desingularization of these singularities, and we have the one-parameter family  $(W_\psi)_{\psi \in \mathbb{P}^1}$  whose fibres are listed as follows:

- When  $\psi$  belongs to  $\mu_5 \subset \mathbb{C} \subset \mathbb{P}^1$ ,  $W_\psi$  has one ordinary double point.
- $W_\infty$  is a normal crossing divisor in the total space.
- The other fibres of  $(W_\psi)_{\psi \in \mathbb{P}^1}$  are smooth with Hodge numbers  $h^{p,q} = 1$  for  $p + q = 3$ ,  $p, q \geq 0$ .

By the action of

$$\alpha \in \mu_5, \quad (x_1, \dots, x_5) \mapsto (x_1, \dots, x_4, \alpha^{-1} x_5),$$

we have the isomorphism from the fibre over  $\psi$  to the fibre over  $\alpha\psi$ . Let  $\lambda = \psi^5$  and let

$$\begin{array}{ccc} (W_\lambda)_{\lambda \in \mathbb{P}^1} & \xlongequal{\quad} & ((W_\psi)_{\psi \in \mathbb{P}^1})/\mu_5 \\ \downarrow & & \downarrow \\ (\lambda\text{-plane}) & \xlongequal{\quad} & (\psi\text{-plane})/\mu_5. \end{array}$$

This family  $(W_\lambda)_{\lambda \in \mathbb{P}^1}$  is the so-called quintic-mirror family. (For more details of the above construction, see e.g. [4], [5].)

### 3. Monodromy

Let  $b \in \mathbb{P}^1 - \{0, 1, \infty\}$  on the  $\lambda$ -plane. In [1], Candelas, de la Ossa, Green and Parks constructed a symplectic basis  $\{A^1, A^2, B_1, B_2\}$  of  $H_3(W_b, \mathbb{Z})$  and calculated the monodromies around  $\lambda = 0, 1, \infty$  on the period integrals of a holomorphic 3-form on this basis. By the relation in [5, Appendix C] between the symplectic basis  $\{\beta^1, \beta^2, \alpha_1, \alpha_2\}$  of  $H^3(W_b, \mathbb{Z})$ , which is defined to be the dual basis of  $\{B_1, B_2, A^1, A^2\}$ , and the period

integrals, we have the matrix representations of the local monodromies for the basis  $\{\beta^1, \beta^2, \alpha_1, \alpha_2\}$ . We recall their results.

Matrix representations  $A, T, T_\infty$  of local monodromies around  $\lambda = 0, 1, \infty$  for the basis  $\{\beta^1, \beta^2, \alpha_1, \alpha_2\}$  are as follows:

$$A = \begin{pmatrix} 11 & 8 & -5 & 0 \\ 5 & -4 & -3 & 1 \\ 20 & 15 & -9 & 0 \\ 5 & -5 & -3 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_\infty = \begin{pmatrix} -9 & -3 & 5 & 0 \\ 0 & 1 & 0 & 0 \\ -20 & -5 & 11 & 0 \\ -15 & 5 & 8 & 1 \end{pmatrix}.$$

In particular, the above  $A$  and  $T$  are the inverse matrices of the matrices  $A$  and  $T$  in the lists of [1], respectively.

Let  $\langle, \rangle$  be the anti-symmetric bilinear form on  $H^3(W_b, \mathbb{Z})$  defined by the cup product. The global monodromy  $\Gamma$  is  $\text{Im}(\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}) \rightarrow \text{Aut}(H^3(W_b, \mathbb{Z}), \langle, \rangle))$ . When we take  $\{\beta^1, \beta^2, \alpha_1, \alpha_2\}$  as the basis of  $H^3(W_b, \mathbb{Z})$ ,  $\text{Aut}(H^3(W_b, \mathbb{Z}), \langle, \rangle)$  is identified with  $\text{Sp}(4, \mathbb{Z})$ , and  $\Gamma$  is the subgroup of  $\text{Sp}(4, \mathbb{Z})$  which is generated by  $A$  and  $T$ .

We can partially normalize  $A$  and  $T$  simultaneously as follows.

**Lemma.** *There exists  $P \in \text{GL}(4, \mathbb{Q})$  such that*

$$P^{-1}A^{-1}P = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 5 & 5 & 5 & -4 \end{pmatrix}, \quad P^{-1}T^{-1}P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof. We take  $P = \begin{pmatrix} 5 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 10 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ . The assertion follows.  $\square$

#### 4. Main result

Let  $\Gamma' = \{P^{-1}XP \in \text{GL}(4, \mathbb{Z}) \mid X \in \Gamma\}$ , and let  $\rho: \text{GL}(4, \mathbb{Z}) \rightarrow \text{GL}(4, \mathbb{Z}/5\mathbb{Z})$  be the natural projection. Define  $\tilde{\Gamma} = \rho(\Gamma')$ . We will study  $\tilde{\Gamma}$ .

Let  $\tilde{A} = \rho(P^{-1}A^{-1}P)$ ,  $\tilde{T} = \rho(P^{-1}T^{-1}P) \in \text{GL}(4, \mathbb{Z}/5\mathbb{Z})$ . By a simple calculation, we obtain

$$\tilde{A}^n = \begin{pmatrix} 1 & n & 3n(n+4) & n(n+1)(4n+1) \\ 0 & 1 & n & 2n(n+1) \\ 0 & 0 & 1 & 4n \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \text{GL}(4, \mathbb{Z}/5\mathbb{Z}).$$

Let  $\hat{\Gamma}$  be

$$\left\{ \begin{pmatrix} 1 & n & 3n^2 + 2n & a \\ 0 & 1 & n & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \text{GL}(4, \mathbb{Z}/5\mathbb{Z}) \mid n, a, b, c \in \mathbb{Z}/5\mathbb{Z} \right\}.$$

$\hat{\Gamma}$  is a subgroup of  $\text{GL}(4, \mathbb{Z}/5\mathbb{Z})$  which contains  $\tilde{A}$  and  $\tilde{T}$ . The following Theorem and Corollary are the main results of this paper.

**Theorem.**  $\tilde{\Gamma} = \hat{\Gamma}$ .

Proof.  $\tilde{\Gamma} \subset \hat{\Gamma}$  follows from what we just mentioned. So we shall prove the converse inclusion.

From the presentations of elements of  $\hat{\Gamma}$ , we see that  $\hat{\Gamma}$  is generated by

$$\tilde{A}, \tilde{T}, E_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, it is enough to show  $E_1$  and  $E_2$  belong to  $\tilde{\Gamma}$ . In fact, we have

$$E_2 = \tilde{A}\tilde{T}\tilde{A}^4\tilde{T}^4, \quad E_1 = (E_2^2\tilde{A}^2\tilde{T}^4\tilde{A}^3\tilde{T})^4.$$

Hence  $E_1, E_2 \in \tilde{\Gamma}$ . □

**Corollary.** *Let  $X \in \Gamma$ . Then the characteristic polynomial of  $X$  is*

$$x^4 + (5m + 1)x^3 + (5n + 1)x^2 + (5m + 1)x + 1,$$

where  $m, n$  are some integers. In particular, if  $X$  is not the unit matrix and the order of  $X$  is finite, then the order of  $X$  is 5 and the eigenvalues of  $X$  are  $\exp(2\pi i/5)$ ,  $\exp(4\pi i/5)$ ,  $\exp(6\pi i/5)$ ,  $\exp(8\pi i/5)$ .

Proof. We shall prove the first part. Let  $\lambda_1, \lambda_2, \lambda_3, \lambda_4$  be the eigenvalues of  $X$ . Then the characteristic polynomial  $p(X)$  of  $X$  is

$$x^4 - \left( \sum_{1 \leq i \leq j \leq k \leq 4} \lambda_i \lambda_j \lambda_k \right) x^3 + \left( \sum_{1 \leq i \leq j \leq 4} \lambda_i \lambda_j \right) x^2 - \left( \sum_{1 \leq i \leq 4} \lambda_i \right) x + 1.$$

On the other hand, the the characteristic polynomial  $p(X^{-1})$  of  $X^{-1}$  is

$$\begin{aligned} x^4 - \left( \sum_{1 \leq i \leq j \leq k \leq 4} \frac{1}{\lambda_i} \frac{1}{\lambda_j} \frac{1}{\lambda_k} \right) x^3 + \left( \sum_{1 \leq i \leq j \leq 4} \frac{1}{\lambda_i} \frac{1}{\lambda_j} \right) x^2 - \left( \sum_{1 \leq i \leq 4} \frac{1}{\lambda_i} \right) x + 1 \\ = x^4 - \left( \sum_{1 \leq i \leq 4} \lambda_i \right) x^3 + \left( \sum_{1 \leq i \leq j \leq 4} \lambda_i \lambda_j \right) x^2 - \left( \sum_{1 \leq i \leq j \leq k \leq 4} \lambda_i \lambda_j \lambda_k \right) x + 1. \end{aligned}$$

Since  $X \in \text{Sp}(4, \mathbb{Z})$ ,  $p(X) = p(X^{-1})$ . So  $p(X)$  is the form  $x^4 + ax^3 + bx^2 + ax + 1$ , where  $a, b \in \mathbb{Z}$ . It follows from the theorem that  $a \equiv -4$ ,  $b \equiv 6 \pmod{5}$ . Hence the claim of the first part follows.

Next we shall prove the latter part. Let  $\lambda$  be an eigenvalue of  $X$ . It follows from  $p(X) = p(\bar{X})$  and  $p(X) = p(X^{-1})$  that  $\bar{\lambda}$ ,  $1/\lambda$ ,  $1/\bar{\lambda}$  are also eigenvalues of  $X$ . Since the determinant of  $X$  is 1, if 1 or  $-1$  is an eigenvalue of  $X$ , its multiplicity is even. Since the order of  $X$  is finite, we can express eigenvalues of  $X$  by  $\exp(i\theta_1)$ ,  $\exp(-i\theta_1)$ ,  $\exp(i\theta_2)$ ,  $\exp(-i\theta_2)$  ( $0 \leq \theta_1, \theta_2 \leq \pi$ ). Then the characteristic polynomial of  $X$  is

$$x^4 - 2(\cos \theta_1 + \cos \theta_2)x^3 + 2(\cos(\theta_1 + \theta_2) + \cos(\theta_1 - \theta_2) + 1)x^2 - 2(\cos \theta_1 + \cos \theta_2)x + 1.$$

By the claim of the first part of this Corollary, we have

$$-2(\cos \theta_1 + \cos \theta_2) = 5m + 1, \quad 2(\cos(\theta_1 + \theta_2) + \cos(\theta_1 - \theta_2) + 1) = 5n + 1, \quad m, n \in \mathbb{Z}.$$

By the addition theorem for cosines, we have

$$2(\cos \theta_1 + \cos \theta_2) = -5m - 1, \quad 4 \cos \theta_1 \cos \theta_2 = 5n - 1.$$

It follows from  $-4 \leq 2(\cos \theta_1 + \cos \theta_2) \leq 4$  that  $m = 0$  or  $-1$ . If  $m = -1$ , then  $\cos \theta_1$ ,  $\cos \theta_2 = 1$  and all eigenvalues of  $X$  are 1. Since the order of  $X$  is finite,  $X$  is the unit matrix. This contradicts the assumption that  $X$  is not the unit matrix. Hence  $m = 0$  and

$$\cos \theta_1 + \cos \theta_2 = -\frac{1}{2}.$$

It follows from  $-4 \leq 4 \cos \theta_1 \cos \theta_2 \leq 4$  that  $n = 0$  or  $1$ . If  $n = 1$ , then  $\cos \theta_1 = \pm 1$ ,  $\cos \theta_2 = \pm 1$ . This contradicts the fact that  $\cos \theta_1 + \cos \theta_2 = -1/2$ . Hence  $n = 0$  and

$$\cos \theta_1 \cos \theta_2 = -\frac{1}{4}.$$

Combining these two equations, we have

$$\cos^2 \theta_1 + \frac{1}{2} \cos \theta_1 - \frac{1}{4} = 0.$$

When we solve this equation for  $\cos \theta_1$ ,

$$\begin{aligned}\cos \theta_1 &= \frac{-1 \pm \sqrt{5}}{4}, & \sin \theta_1 &= \frac{\sqrt{10 \pm 2\sqrt{5}}}{4}, \\ \cos \theta_2 &= \frac{-1 \mp \sqrt{5}}{4}, & \sin \theta_2 &= \frac{\sqrt{10 \mp 2\sqrt{5}}}{4}.\end{aligned}$$

Then we can verify easily that  $(\exp(i\theta_1))^5$  and  $(\exp(i\theta_2))^5 = 1$ . Hence  $(\theta_1, \theta_2) = (2\pi/5, 4\pi/5)$  or  $(4\pi/5, 2\pi/5)$ .  $\square$

## 5. A relation to the other result

In this section, we shall compare the main result of this paper with the result of Chen, Yang and Yui. In [2], they find the congruence subgroup  $\Gamma(5, 5)$  which contains the global monodromy  $\Gamma$ . Combining their result and our theorem, we can find a smaller group which contains  $\Gamma$ .

The congruence subgroup  $\Gamma(5, 5)$  is defined by

$$\Gamma(5, 5) = \left\{ X \in \mathrm{Sp}(4, \mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{pmatrix} \pmod{5} \right\}.$$

$\Gamma(5, 5)$  contains the principal congruence group  $\Gamma(5) = \mathrm{Ker}(\mathrm{Sp}(4, \mathbb{Z}) \rightarrow \mathrm{Sp}(4, \mathbb{Z}/5\mathbb{Z}))$  as a normal subgroup of finite index.

Let  $X \in \Gamma(5, 5)$  and express  $X$  by

$$\begin{pmatrix} 5x_{11} + 1 & x_{12} & x_{13} & x_{14} \\ 5x_{21} & 5x_{22} + 1 & x_{23} & x_{24} \\ 5x_{31} & 5x_{32} & 5x_{33} + 1 & 5x_{34} \\ 5x_{41} & 5x_{42} & x_{43} & 5x_{44} + 1 \end{pmatrix}, \quad x_{ij} \in \mathbb{Z} \quad (1 \leq i, j \leq 4).$$

Then we have

$$\mathrm{GL}(4, \mathbb{Z}) \ni P^{-1}XP \equiv \begin{pmatrix} 1 & -9x_{31} & -x_{12} + 3x_{32} & -x_{14} + 3x_{34} \\ 0 & 1 & -2x_{12} & -2x_{14} \\ 0 & 0 & 1 & x_{24} \\ 0 & 0 & 0 & 1 \end{pmatrix} \pmod{5}.$$

By the main theorem, if  $X \in \Gamma$ , then  $\rho(P^{-1}XP) \in \tilde{\Gamma}$  and

$$-9x_{31} \equiv n, \quad -2x_{12} \equiv n, \quad -x_{12} + 3x_{32} \equiv 3n^2 + 2n \pmod{5}.$$

where  $n$  is some integer. From a simple calculation, the above equation is equivalent to

$$x_{31} \equiv 3x_{12}, \quad x_{32} \equiv 4x_{12}^2 + 4x_{12} \pmod{5}.$$

So we define

$$\tilde{\Gamma}(5, 5) = \left\{ \begin{pmatrix} 5x_{11}+1 & x_{12} & x_{13} & x_{14} \\ 5x_{21} & 5x_{22}+1 & x_{23} & x_{24} \\ 5x_{31} & 5x_{32} & 5x_{33}+1 & 5x_{34} \\ 5x_{41} & 5x_{42} & x_{43} & 5x_{44}+1 \end{pmatrix} \in \mathrm{Sp}(4, \mathbb{Z}) \mid \begin{array}{l} x_{31} \equiv 3x_{12}, \\ x_{32} \equiv 4x_{12}^2 + 4x_{12} \\ \pmod{5} \end{array} \right\}.$$

Then we have the following Corollary.

- Corollary.** (i)  $\tilde{\Gamma}(5, 5)$  is a subgroup of  $\Gamma(5, 5)$ .  
(ii)  $\Gamma \subset \tilde{\Gamma}(5, 5) \subsetneq \Gamma(5, 5)$ .  
(iii)  $\tilde{\Gamma}(5, 5)$  is a congruence subgroup of  $\mathrm{Sp}(4, \mathbb{Z})$  of finite index.

*Proof.* Let  $\rho' : \Gamma(5, 5) \rightarrow \mathrm{GL}(4, \mathbb{Z})$ ,  $X \mapsto P^{-1}XP$  and let  $\pi = \rho \circ \rho' : \Gamma(5, 5) \rightarrow \mathrm{GL}(4, \mathbb{Z}/5\mathbb{Z})$ .  $\tilde{\Gamma}(5, 5) = \pi^{-1}(\tilde{\Gamma})$  follows from what we just mentioned. Since  $\pi$  is a group homomorphism,  $\pi^{-1}(\tilde{\Gamma})$  is a subgroup of  $\Gamma(5, 5)$ . Hence the claim of (i) follows.

We can verify easily that  $A$  and  $T$  belong to  $\tilde{\Gamma}(5, 5)$ . Therefore  $\tilde{\Gamma}(5, 5)$  contains  $\Gamma$ .

We shall show that  $\tilde{\Gamma}(5, 5)$  is a proper subgroup of  $\Gamma(5, 5)$ . We take  $X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ .

Then  $X \in \Gamma(5) \subset \Gamma(5, 5)$  and  $X \notin \tilde{\Gamma}(5, 5)$ . Hence the claim of (ii) follows.

Finally, we shall show the claim of (iii).  $\tilde{\Gamma}(5, 5)$  contains the principal congruence subgroup  $\Gamma(25) = \mathrm{Ker}(\mathrm{Sp}(4, \mathbb{Z}) \rightarrow \mathrm{Sp}(4, \mathbb{Z}/25\mathbb{Z}))$  as a normal subgroup. Hence we obtain  $|\tilde{\Gamma}(5, 5) : \mathrm{Sp}(4, \mathbb{Z})| < |\Gamma(25) : \mathrm{Sp}(4, \mathbb{Z})| = |\mathrm{Sp}(4, \mathbb{Z}/25\mathbb{Z})| < \infty$ .  $\square$

**QUESTION.** There are other 13 mirror families of Calabi–Yau threefolds with  $h^{2,1} = 1$  as discussed in [2]. Is it possible to find smaller subgroups in those 13 cases as well?

**ACKNOWLEDGEMENT.** The author would like to thank Professors Sampei Usui, Atsushi Takahashi and Keiji Oguiso for their helpful advices and suggestions. The author would also like to thank the referee for the very helpful comments.

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Department of Mathematics  
Graduate School of Science  
Osaka University  
Osaka 560-0043  
Japan

Current address:  
Healthcare Solution Delivery No. 2  
Global Business Services  
IBM Japan Ltd.  
Osaka 550-0004  
Japan  
e-mail: E35135@jp.ibm.com